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μ Synthesis Using Linear Quadratic Gaussian Controllers

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A sufficient condition is derived for the existence of the optimal μ controllers by showing the equivalence between the optimal μ problem and the weighted H_2-optimization problem. The weighted H_2-optimization technique used in linear quadratic Gaussian design with loop transfer recovery is exploited to generate a sequence of H_2 controllers converging to the optimal μ controller. The resulting optimal μ controller not only has the inherent robust-performance property resulting from the μ criterion, but also possesses the nice H_2 control structure, being easy to compute and implement. A flight control example is demonstrated to show that the numerical accuracy of the H_2-based μ-synthesis approach is comparable to the conventional H_∞-based μ-synthesis technique (*D–K* iteration) but with reduced computational efforts.

Nomenclature

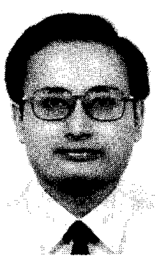
$\|A(s)\|_2$	$= \{(1/2\pi)\int_{-\infty}^{\infty} \mathrm{tr}[A^*(j\omega)A(j\omega)]\,\mathrm{d}\omega\}^{1/2}$
	$= \{(1/2\pi)\int_{-\infty}^{\infty} \sum_i \sigma_i^2[A(j\omega)]\,\mathrm{d}\omega\}^{1/2}$
$\|A(s)\|_\infty$	$= \sup_{-\infty<\omega<\infty} \bar{\sigma}[A(j\omega)]$
$\boldsymbol{B}_\Delta$	$= \{\Delta \in \boldsymbol{S}_\Delta : \bar{\sigma}(\Delta) \le 1\}$
$\boldsymbol{D}$	$= \{\mathrm{diag}[D_1, \ldots, D_s, d_1 I_{m_1}, \ldots, d_f I_{m_f}] : D_i \in \boldsymbol{C}^{r_i \times r_i}, D_i = D_i^* > 0, d_j \in \boldsymbol{R}^+\}$
K_i	= optimal H_2 controller at the ith iteration
M	$= F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$
P	= augmented plant, $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$
$\boldsymbol{Q}$	$= \{Q \mid Q^*Q = I_n, Q \in \boldsymbol{S}_\Delta\}$
$\boldsymbol{S}_\Delta$	$= \{\mathrm{diag}[\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}, \Delta_1, \ldots, \Delta_f] : \delta_i \in \boldsymbol{C}, \Delta_j \in \boldsymbol{C}^{m_j \times m_j}\}$
$W_i(s)$	= scalar weighting function at the ith iteration
$\gamma_i(\omega)$	$= \mu_\Delta[W_i F_l(P, K_i)(j\omega)]$
$\zeta_i(\omega)$	$= \mu_\Delta[F_l(P, K_i)(j\omega)]$
λ_i	$= \|W_i F_l(P, K_i)\|_2$
$\mu_\Delta(M)$	$= 1/\min_\Delta\{\bar{\sigma}(\Delta) : \Delta \in \boldsymbol{S}_\Delta, \det(I - M\Delta) = 0\}$
	$= 0$ if $\det(I - M\Delta) \ne 0, \forall \Delta \in \boldsymbol{S}_\Delta$
$\rho(M)$	= spectral radius of M
$\bar{\sigma}(M)$	= largest singular value of M
$\sigma_i(M)$	= ith largest singular value of M

Introduction

MAINTAINING an acceptable level of performance, as well as stability, in the presence of disturbances and model uncertainties, namely, the robust-performance problem, has long been recognized as a crucial problem in control system design. Recently, by introducing the structured singular value μ as a measure of performance robustness, Doyle[1] proposed a celebrated framework for analyzing the robust-performance problem and suggested a synthesis procedure for designing robust-performance controllers called μ synthesis.

The solution to the μ-synthesis problem requires minimization of the structured singular value μ of a cost function matrix, which is still an open problem and as yet has not been analytically solved. A numerical approximation solution, the *D–K* iteration, was established by Doyle,[2] where the μ minimization is approximated by a sequence of H_∞ minimizations. Though *D–K* iteration does not guarantee the convergence to the optimal μ controller, the algorithm does prove itself as an efficient way of synthesizing robust-performance controllers. Most recently, several alternatives to *D–K* iteration have been proposed. These alternatives include μ–*K* iteration,[3] *E–K* iteration,[4] and *L–R* iteration.[5]

A general framework for feedback control systems is shown in Fig. 1. It is known that any linear interconnection of inputs, outputs, disturbances, and model uncertainties can be recast into this



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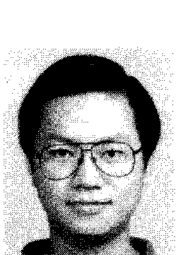

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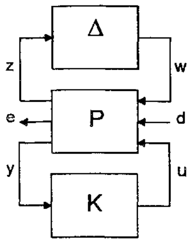

Fig. 1 General framework of feedback control system.

diagram, using linear fractional transformations. In the scenario of Fig. 1, P represents the augmented plant, Δ is the model uncertainty, K is the controller to be designed, $\boldsymbol{d}$ is a vector consisting of exogeneous inputs and disturbances, and $\boldsymbol{e}$ is a vector of penalized signals. The uncertainties Δ, the exogeneous inputs $\boldsymbol{d}$, and the error signal $\boldsymbol{e}$ are normalized to unity. This requires that all scalings and weighting functions should be absorbed in the augmented plant P. Let M represent the lower linear fractional transformation of P closed by K, that is, $M = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$. Obviously, the structured singular value $\mu_\Delta(M)$ depends on both matrix M and the structure of Δ. The main difference between H_∞ design and μ synthesis is the structure of the uncertainty block Δ. For H_∞ design, Δ is a full block; whereas for μ synthesis, Δ is of mixed structure having both scalar and full blocks. We list several properties of the structured singular value, which will be referred to later.

Lemma 1[1,2,6]:

$$\mu_\Delta(\alpha G) = |\alpha|\mu_\Delta(G), \qquad \forall \alpha \in C \tag{1}$$

$$\mu_\Delta(AB) \le \|A\|_\infty \mu_\Delta(B) \tag{2}$$

$$\rho(M) \le \mu_\Delta(M) \le \bar{\sigma}(M) \tag{3}$$

$$\max_{Q \in \boldsymbol{Q}} \rho(QM) \le \mu_\Delta(M) \le \inf_{D \in \boldsymbol{D}} \bar{\sigma}(DMD^{-1}) \tag{4}$$

It is known that the upper bound in Eq. (3) is achieved when Δ is a full block (i.e., unstructured uncertainty) and the lower bound in Eq. (3) is achieved when Δ is a scalar block. The introduction of the scaling factors D and Q in Eq. (4) is to reduce the gap between $\mu_\Delta(M)$ and its upper and lower bounds, respectively.

Robust performance is said to be attained if and only if $\mu_\Delta[F_l(P, K)(j\omega)]$ with $\Delta \in \boldsymbol{B}_\Delta$ satisfies $\mu_\Delta[F_l(P, K)(j\omega)] < 1$, $\forall\omega$. The optimal robust-performance problem (μ systhesis) is to find controller K that stabilizes the augmented plant P and minimizes the worst-case value of $\mu_\Delta[F_l(P, K)(j\omega)]$, i.e.,

$$\inf_{K \text{ stabilizing}} \sup_{\omega \in \boldsymbol{R}} \mu_\Delta[(F_l(P, K)(j\omega)] \tag{5}$$

Because of the inherent nonconvexity of the problem, the complete solutions to this problem are still unavailable. All of the existing numerical schemes solve the optimization problem indirectly. D–K iteration starts from the upper bound of $\mu_\Delta[F_l(P, K)(j\omega)]$. Instead of minimizing $\mu_\Delta[F_l(P, K)(j\omega)]$, D–K iteration minimizes its upper bound $\inf_{D \in \boldsymbol{D}} \bar{\sigma}(DMD^{-1})$, as shown in Eq. (4), and reduces the μ synthesis to the following two-parameter optimization problem:

$$\inf_{K \text{stabilizing}} \sup_{\omega \in \boldsymbol{R}} \inf_{D \in \boldsymbol{D}} \bar{\sigma}\left[DF_l(P, K)D^{-1}(j\omega)\right] \tag{6}$$

This problem can be solved approximately by solving first for K keeping D constant and then by solving for D keeping K constant and so on. For a fixed D, Eq. (6) is an H_∞-optimization problem; whereas for fixed K, Eq. (6) is a convex optimization problem in D. The scaling factors $D_0, D_1, \ldots, D_{i-1}$ obtained before the ith iteration should be multiplied together to obtain the H_∞ controller at the ith iteration.

μ Synthesis and Weighted H_2 Optimization

The motivation of the present research stems from the concept of linear quadratic Gaussian (LQG) design with loop transfer recovery (LTR). It is well recognized that the performance robustness of the LQG controllers can be remarkably improved by the LTR technique. We are curious to know whether the performance robustness of LQG can be ultimately increased by the LTR to the level achieved by the optimal μ controllers. The answer is affirmative, and a workable scheme is proposed in the following to realize this conjecture. The significance of this approach is that the resulting optimal μ controllers can be assigned a priori with LQG structure. First, we will derive a sufficient condition based on weighted H_2 optimization to guarantee the existence of the optimal μ controller.

Theorem 1: Suppose there exists a scalar frequency-dependent weight $W(s)$ such that the optimal H_2 controller K_0 obtained from

$$K_0 = \arg\inf_K \|WF_l(P, K)\|_2 \tag{7}$$

satisfies

$$\mu_\Delta[F_l(P, K_0)(j\omega)] = \gamma_0 = \text{const}, \qquad \forall\omega \tag{8}$$

$$\mu_\Delta[F_l(P, K_0)(j\omega)] = \|WF_l(P, K_0)\|_2, \qquad \forall\omega \tag{9}$$

then K_0 is the optimal μ controller

$$\inf_K \sup_\omega \mu_\Delta[F_l(P, K)(j\omega)] = \mu_\Delta[F_l(P, K_0)(j\omega)] = \gamma_0, \quad \forall\omega \tag{10}$$

Proof: This can be proved easily by contradiction. Suppose K_0' is another controller satisfying Eqs. (8) and (9) but with a smaller μ value:

$$\mu_\Delta[F_l(P, K_0')(j\omega)] = \mu_0' < \mu_0 = \mu_\Delta[F_l(P, K_0)(j\omega)], \qquad \forall\omega \tag{11}$$

Using Eq. (9), Eq. (11) can be rewritten as

$$\|WF_l(P, K_0')\|_2 < \|WF_l(P, K_0)\|_2$$

but this result contradicts the assumption of Eq. (7). Hence, we must have

$$K_0 = \arg\inf_K \sup_\omega \mu_\Delta[F_l(P, K)(j\omega)] = \arg\inf_K \|WF_l(P, K)\|_2 \tag{12}$$

QED

Theorem 1 reveals the fact that an optimal H_2 controller obtained from Eq. (7), satisfying the conditions in Eqs. (8) and (9), is also an optimal μ controller. The all-pass property (8) is an important characteristic of the optimal μ control, and many other optimization problems possess this property.[7] However, the conditions given in Eqs. (7–9) are only sufficient for the existence of the optimal μ controllers, i.e., Theorem 1 does not imply that an optimal μ controller must possess the all-pass property or can always be derived from the weighted H_2 optimization (12). The key issue in Theorem 1 is the characterization of the weighting function $W(s)$. The next section is devoted to characterizing the desired $W(s)$ such that Eq. (12) is satisfied. The right-hand side of Eq. (12) is known as the frequency-weighted LQG problem in the literature,[8–10] which is intimately related to the LQG/LTR technique.[11] Equation (12) reveals the concept that the μ-optimization problem is equivalent to the frequency-weighted H_2-optimization problem. To the authors' knowledge, this concept has not been discussed in the literature to date.

The optimal μ controller K_0 obtained from the weighted H_2-optimization process not only has the inherent robust-performance property because of the μ criterion but also possesses the nice H_2 control structure, being easy to compute and implement.

μ Synthesis via H_2-Based Loop-Shaping Design

In this section we propose a loop-shaping design procedure to construct the weighting function W and the corresponding optimal μ controller in Eq. (12) for arbitrary structure of $F_l(P, K)$. The ultimate closed-loop shape we want to achieve is a uniform frequency response of the structured singular value of $F_l(P, K)$, i.e., the loop shape represented by the all-pass condition (8). The strategy we will adopt is to shape W recursively by a sequence of H_2-optimization processes until a uniform shape of $\mu_\Delta[F_l(P, K)(j\omega)]$ is achieved.

The underlying concept of the H_2-based loop-shaping design comes from the observation of Eq. (9). An admissible $W(s)$ satisfying condition (9) has the following property:

$$|W(j\omega)| = \frac{\mu_\Delta[WF_l(P,K)(j\omega)]}{\|WF_l(P,K)\|_2} \tag{13}$$

where the property $\mu_\Delta[WF_l(P,K)(j\omega)] = |W(j\omega)|\mu_\Delta[F_l(P,K)\times(j\omega)]$ from Eq. (1) has been used. Because both $W(s)$ and $K(s)$ are unknown, we can neither determine $W(s)$ from Eq. (13) nor determine $K(s)$ from the optimization of $\|WF_l(P,K)\|_2$ in Eq. (7); however, if we apply Eqs. (7) and (13) to different iteration steps, $K(s)$ and $W(s)$ can be determined iteratively from each other. For example, if we have a sequence of optimal H_2 controllers $(K_0, K_1, \ldots, K_i, \ldots)$, where we suppose controllers $K_0(s)$–$K_{i-1}(s)$ and the corresponding frequency-dependent weights $W_0(s)$–$W_{i-1}(s)$ are known, the objective is to determine $K_i(s)$ and $W_i(s)$. We can exploit $K_{i-1}(s)$ and $W_{i-1}(s)$ to predict $W_i(s)$ by applying Eq. (13):

$$|W_i(j\omega)| = \frac{\mu_\Delta[W_{i-1}F_l(P,K_{i-1})(j\omega)]}{\|W_{i-1}F_l(P,K_{i-1})\|_2} \tag{14}$$

Next, we use this $W_i(s)$ to determine the optimal H_2 controller K_i from Eq. (7):

$$K_i = \arg \inf_{K\,\text{stabilizing}} \|W_iF_l(P,K)\|_2 \tag{15}$$

In general, W_i obtained from Eq. (14) may not satisfy Eq. (13), but we expect that the limit of the sequence, W_∞, can satisfy Eq. (13).

As the iteration proceeds, the following sequences can be defined:

$$\lambda_i = \|W_iF_l(P,K_i)\|_2 \tag{16}$$

$$\zeta_i(\omega) = \mu_\Delta[F_l(P,K_i)(j\omega)] \tag{17}$$

$$\gamma_i(\omega) = \mu_\Delta[W_iF_l(P,K_i)(j\omega)] \tag{18}$$

where $W_i(s)$ is a scalar, minimum-phase transfer function whose magnitude is determined by $\gamma_{i-1}(\omega)/\lambda_{i-1}$, that is,

$$|W_i(j\omega)| = \frac{\gamma_{i-1}(\omega)}{\lambda_{i-1}}, \qquad i = 1,2,\ldots \tag{19}$$

with $|W_0(j\omega)| = 1$. The sequence of optimal H_2 controllers converging to an optimal μ controller can be characterized in the following way.

Theorem 2: Given the controller sequence $(K_i)_{i=0}^\infty$ defined by

$$K_i = \arg \inf_{K\,\text{stabilizing}} \|W_iF_l(P,K)\|_2 \tag{20}$$

with W_i given by Eq. (19), then the corresponding sequences $(\lambda_i)_{i=0}^\infty$, $[\zeta_i(\omega)]_{i=0}^\infty$, and $[\gamma_i(\omega)]_{i=0}^\infty$, possess the following properties.

Property 1:

$$\|\gamma_i(\omega)\|_2 \le \lambda_i \tag{21}$$

Property 2:

$$\frac{\zeta_{i+1}(\omega)}{\lambda_i}\gamma_i(\omega) = \gamma_{i+1}(\omega), \qquad \forall\omega \tag{22}$$

Property 3:

$$|W_{i+1}(j\omega)| = \frac{\gamma_i(\omega)}{\lambda_i} = \prod_{k=0}^{i}\frac{\zeta_k(\omega)}{\lambda_k}, \qquad \forall\omega \tag{23}$$

Proof:

1) Exploiting the relation

$$\mu_\Delta^2[W_iF_l(P,K_i)(j\omega)] \le \bar\sigma^2[W_iF_l(P,K_i)(j\omega)] \le \sum_{k=1}^{n}\sigma_k^2[W_iF_l(P,K_i)(j\omega)]$$

and integrating with respect to ω from $-\infty$ to ∞, we have

$$\int_{-\infty}^{\infty}\mu_\Delta^2[W_iF_l(P,K_i)(j\omega)]\,d\omega \le \int_{-\infty}^{\infty}\sigma_1^2[W_iF_l(P,K_i)(j\omega)]\,d\omega \le \int_{-\infty}^{\infty}\sum_{k=1}^{n}\sigma_k^2[W_iF_l(P,K_i)(j\omega)]\,d\omega$$

From the definition of the H_2 norm, the preceding equation leads to the desired result,

$$\|\gamma_i\|_2 = \|\mu_\Delta[W_i(j\omega)F_l(P,K_i)(j\omega)]\|_2 \le \|\sigma_1[W_i(j\omega)F_l(P,K_i)(j\omega)]\|_2 \le \|W_iF_l(P,K_i)\|_2 = \lambda_i$$

2) By definition,

$$\gamma_{i+1}(\omega) = \mu_\Delta[W_{i+1}(j\omega)F_l(P,K_{i+1})(j\omega)] = |W_{i+1}(j\omega)|\mu_\Delta[F_l(P,K_{i+1})(j\omega)] = [\gamma_i(\omega)/\lambda_i]\zeta_{i+1}(\omega)$$

3) By applying Eq. (22) repeatedly, we have

$$|W_{i+1}(\omega)| = \frac{\gamma_i(\omega)}{\lambda_i} = \frac{\zeta_i(\omega)}{\lambda_i}\frac{\gamma_{i-1}(\omega)}{\lambda_{i-1}} = \frac{\zeta_i(\omega)}{\lambda_i}\frac{\zeta_{i-1}(\omega)}{\lambda_{i-1}}\frac{\gamma_{i-2}(\omega)}{\lambda_{i-2}} = \cdots = \prod_{k=0}^{i}\frac{\zeta_k(\omega)}{\lambda_k} \qquad \text{QED}$$

Now we are ready to show that the limit controller K_∞ in the sequence of the optimal H_2 controllers $(K_i)_{i=0}^\infty$ defined in Eq. (20) is the solution of the μ-optimization problem (5). The proof contains three steps.

1) The first step is to show the convergence of the controller sequence $(K_i)_{i=0}^\infty$.

2) The second step is to show that the limit controller K_∞ satisfies the all-pass condition (8).

3) The third step is to show that the limit controller K_∞ truly achieves the infimum:

$$\inf_K \sup_\omega \mu_\Delta[F_l(P,K)(j\omega)] = \mu_\Delta[F_l(P,K_\infty)(j\omega)] = \zeta_\infty, \qquad \forall\omega$$

The convergence of the controller sequence $(K_i)_{i=0}^\infty$ is proved first.

Theorem 3:

1) Let K_i be the central solution[12] of the H_2-optimization problem defined in Eq. (20), then

$$K_i = \arg\inf_K \left\|\prod_{k=0}^{i-1}\frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty}F_l(P,K)(j\omega)\right\|_2 \tag{24}$$

2) The sequence $(\bar\lambda_i)_{i=0}^\infty$ is convergent, where $\bar\lambda_i$ is defined as

$$\bar\lambda_0 = \inf_K\|F_l(P,K)\|_2$$
$$\bar\lambda_i = \inf_K \left\|\prod_{k=0}^{i-1}\frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty}F_l(P,K)(j\omega)\right\|_2, \quad i = 1,2,3,\ldots \tag{25}$$

3) The controller sequence $(K_i)_{i=0}^\infty$ is convergent.

Proof:

1) The definition of K_i is given in Eq. (20). Exploiting Eq. (23), we can rewrite K_i as

$$\begin{aligned} K_i &= \arg\inf_K \| W_i F_l(P, K)(j\omega)\|_2 \\ &= \arg\inf_K \left\| \prod_{k=0}^{i-1} \frac{\zeta_k(\omega)}{\lambda_k} F_l(P, K)(j\omega)\right\|_2 \\ &= \arg \left(\prod_{k=0}^{i-1} \frac{\|\zeta_k\|_\infty}{\lambda_k} \inf_K \left\| \prod_{k=0}^{i-1} \frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty} F_l(P, K)(j\omega)\right\|_2 \right) \\ &= \arg\inf_K \left\| \prod_{k=0}^{i-1} \frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty} F_l(P, K)(j\omega)\right\|_2 \end{aligned}$$

where we have used the fact that both $\|\zeta_k(\omega)\|_\infty$ and λ_k are independent of frequency ω.

2) By definition,

$$\begin{aligned} \bar{\lambda}_i &= \left\| \prod_{k=0}^{i-1} \frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty} F_l(P, K_i)(j\omega)\right\|_2 \\ &= \left\| \frac{\zeta_i(\omega)}{\|\zeta_i\|_\infty}\right\|_\infty \left\| \prod_{k=0}^{i-1} \frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty} F_l(P, K_i)\right\|_2, \\ &\qquad \left(\text{note } \left\| \frac{\zeta_i(\omega)}{\|\zeta_i\|_\infty}\right\|_\infty = 1\right) \\ &\geq \left\| \prod_{k=0}^{i} \frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty} F_l(P, K_i)\right\|_2 \\ &\geq \inf_K \left\| \prod_{k=0}^{i} \frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty} F_l(P, K)\right\|_2 = \bar{\lambda}_{i+1} \end{aligned}$$

Hence, the sequence $(\bar{\lambda}_i)_{i=0}^\infty$ is monotonically decreasing and bounded by $0 \leq \bar{\lambda}_i \leq \bar{\lambda}_0$. Employing the convergent theorem[13] of real sequences, we thus obtain the convergence of $(\bar{\lambda}_i)_{i=0}^\infty$.

3) According to the definition of $\bar{\lambda}_i$, we know that $\bar{\lambda}_i$ is a continuous function of K_i. If we let K_i be the central controller developed in Ref. 12, then K_i is uniquely determined by $\bar{\lambda}_i$. Since $\bar{\lambda}_i$ is a one-to-one continuous function of K_i, the convergence of the sequence $(\bar{\lambda}_i)_{i=0}^\infty$ implies the convergence of the controller sequence $(K_i)_{i=0}^\infty$. QED

Having proved the convergence of the controller sequence K_i, we proceed to verify the uniformity of the structured singular value of $F_l(P, K_\infty)(j\omega)$.

Lemma 2: Let K_∞ be the limit of the controller sequence defined in Eq. (20), then

$$\mu_\Delta[F_l(P, K_\infty)(j\omega)] = \lambda_\infty = \text{const}, \qquad \forall\omega \tag{26}$$

Proof: The three sequences defined in Eqs. (16–18) are determined uniquely by the controller sequence K_i. Therefore, the convergence of K_i implies the convergence of λ_i, $\zeta_i(\omega)$, and $\gamma_i(\omega)$. On the other hand, from Eq. (22) we have

$$\lim_{i\to\infty} \frac{\zeta_i(\omega)}{\lambda_{i-1}} = \lim_{i\to\infty} \frac{\gamma_i(\omega)}{\gamma_{i-1}(\omega)}$$

Since $\gamma_i(\omega)$ is convergent, we have $\lim_{i\to\infty} \gamma_i(\omega)/\gamma_{i-1}(\omega) = 1$, $\forall\omega$. It turns out that $\lim_{i\to\infty} \zeta_i(\omega)/\lambda_{i-1} = 1$, $\forall\omega$, that is, $\zeta_\infty(\omega) = \mu_\Delta[F_l(P, K_\infty)(j\omega)] = \lambda_\infty = \text{const}$, $\forall\omega$. QED

The final step in our characterization of the optimal μ controller via H_2-based loop shaping design is to show that K_∞ achieves the infimum of μ.

Theorem 4: The limit controller K_∞ of the controller sequence defined in Eq. (20) achieves the infimum:

$$K_\infty = \arg\inf_K \sup_\omega \mu_\Delta[F_l(P, K)(j\omega)] = \arg\inf_K \|W_\infty F_l(P, K)\|_2 \tag{27}$$

Proof: From the definition of W_i in Eq. (19), the convergence of W_i is guaranteed by the convergence of $\gamma_i(\omega)$ and λ_i. Let the limit of the sequence $(W_i)_{i=0}^\infty$ be denoted by W_∞. Then Eq. (14) becomes

$$\begin{aligned} |W_\infty(j\omega)| &= \frac{\mu_\Delta[W_\infty F_l(P, K_\infty)(j\omega)]}{\|W_\infty F_l(P, K_\infty)\|_2} \\ &= |W_\infty(j\omega)| \frac{\mu_\Delta[F_l(P, K_\infty)(j\omega)]}{\|W_\infty F_l(P, K_\infty)\|_2} \end{aligned}$$

where the property from Eq. (1) has been used. Then, we have

$$\mu_\Delta[F_l(P, K_\infty)(j\omega)] = \|W_\infty F_l(P, K_\infty)(j\omega)\|_2 \tag{28}$$

where by the definition W_∞ is related to K_∞ via

$$K_\infty = \arg\inf_K \|W_\infty F_l(P, K)(j\omega)\|_2 \tag{29}$$

Keeping the results of Eqs. (26), (28), and (29) in mind, we go back to Theorem 1 where we can see that the conditions (7–9) are satisfied by the controller K_∞ and the weighting function W_∞. Hence, K_∞ is the desired optimal μ controller. QED

Remarks:

1) No particular structure of the uncertainty has been assumed in the process of proof. Hence, the success of the μ synthesis using the aforementioned H_2-based loop-shaping design is independent of the structure of the uncertainties, which can be real, complex, or mixed.

2) The inherent assumption made in Theorems 3 and 4 is that the structured singular value $\mu_\Delta(M)$ of a given constant matrix $M = F_l(P, K)(j\omega_0)$ with given uncertainty structure Δ can be evaluated perfectly. However, the existing numerical schemes can only calculate a tight upper bound of $\mu_\Delta(M)$ such as $\inf_{D\in\mathbf{D}} \bar{\sigma}(DMD^{-1})$ [see Eq. (4)], and an exact evaluation of $\mu_\Delta(M)$ is still lacking in the literature except for some special cases where $2s + f \leq 3$ with s and f being the numbers of the repeated scalar blocks and the full blocks, respectively. Therefore, although the convergence of the H_2 controller sequence $(K_i)_{i=0}^\infty$ to the optimal μ controller is theoretically guaranteed, the exact calculation of K_i is, in general, not available using the current art of structured singular-value computation.

3) If the computation of $\mu_\Delta(M)$ is replaced with its upper bound $\inf_{D\in\mathbf{D}} \bar{\sigma}(DMD^{-1})$ and if $2s + f \leq 3$ is satisfied, $(K_i)_{i=0}^\infty$ still converges to the optimal μ controller; whereas if $2s + f > 3$, then $(K_i)_{i=0}^\infty$ only converges to an approximation of the optimal μ controller. However, if the computation of $\mu_\Delta(M)$ is replaced with its lower bound $\max_{Q\in\mathbf{Q}} \rho(QM)$, the resulting $(K_i)_{i=0}^\infty$ converges exactly to the optimal μ controller, since $\mu_\Delta(M) = \max_{Q\in\mathbf{Q}} \rho(QM)$. Nevertheless, note that the computation of $\max_{Q\in\mathbf{Q}} \rho(QM)$ is not a convex optimization problem.

H_2-Based μ-Synthesis Algorithm

Up to now we have developed the theoretical background for the synthesis of optimal μ controllers via an H_2-based loop-shaping procedure. The equivalence in Eq. (12) is established by the controller K_∞ and the weighting function W_∞, where K_∞ and W_∞ are the limits of the sequences $(K_\infty)_{i=0}^\infty$ and $(W_\infty)_{i=0}^\infty$, respectively. Using Eq. (23), the controller sequence $(K_i)_{i=0}^\infty$ defined in Eq. (20) can be rewritten in the following manner:

$$\begin{aligned} K_i &= \arg\inf_K \left\| \prod_{k=0}^{i-1} \frac{\zeta_k(\omega)}{\lambda_k} F_l(P, K)(j\omega)\right\|_2 \\ &= \arg\inf_K \left\| \prod_{k=0}^{i-1} \zeta_k(\omega) F_l(P, K)(j\omega)\right\|_2 \end{aligned} \tag{30}$$

If we define a new scalar minimum-phase function $\hat{W}_i$ as

$$|\hat{W}_i(j\omega)| = \prod_{k=0}^{i-1} \zeta_k(\omega)$$

then Eq. (30) becomes $K_i = \arg\inf_K \|\hat{W}_i F_l(P, K)\|_2$. Instead of the frequency-dependent weight $W_i(s)$ defined in Eq. (19), we can use $\hat{W}_i(s)$ as a new frequency-dependent weight. The following work is to derive the recursive formula for $\hat{W}_i$:

$$|\hat{W}_{i+1}(j\omega)| = \prod_{k=0}^{i} \zeta_k(\omega) = \left[\prod_{k=0}^{i-1} \zeta_k(\omega)\right]\zeta_i(\omega)$$
$$= |\hat{W}_i(j\omega)|\mu_\Delta[F_l(P, K_i)(j\omega)]$$
$$= \mu_\Delta[\hat{W}_i F_l(P, K_i)(j\omega)] \tag{31}$$

In terms of the new frequency-dependent weight $\hat{W}_i(s)$, we summarize the μ-synthesis technique using H_2-based loop-shaping procedures in the following algorithm.

Initialization: $i = 0$.

1) Set $\hat{W}_0(s) = 1$.

2) Compute $K_0 = \arg\inf_K \|\hat{W}_0 F_l(P, K)\|_2$ using the H_2-optimization technique.

3) Compute $\mu_\Delta[\hat{W}_0 F_l(P, K_0)(j\omega)]$.

Recursive formula: $i = 1, 2, 3, \ldots$.

1) Set $\deg(\hat{W}_i) = n_w$, where $\hat{W}_i$ is a scalar, minimum-phase function.

2) Fit $\mu_\Delta[\hat{W}_{i-1} F_l(P, K_{i-1})(j\omega)]$ by $|\hat{W}_i(j\omega)|$ [from Eq. (31)].

3) Compute $K_i = \arg\inf_K \|\hat{W}_i F_l(P, K)\|_2$.

4) Compute $\mu_\Delta[\hat{W}_i F_l(P, K_i)(j\omega)]$.

Repeat the algorithm until the required accuracy in the uniformity of $\zeta_i(\omega) = \mu_\Delta[F_l(P, K_i)(j\omega)]$ is met. This algorithm is a new scheme for μ synthesis. Ideally, at the end of the iteration, $\mu_\Delta[F_l(P, K_\infty)(j\omega)]$ must be an all-pass function, as is shown in Lemma 2. Although the convergence of the sequence to the optimal μ controller is guaranteed theoretically, in numerical calculation, the achievable degree of optimization depends on the accuracy in calculating the optimal H_2 controller K_i and the structured singular value $\mu_\Delta[\hat{W}_{i-1} F_l(P, K_{i-1})(j\omega)]$, and on the accuracy of curve fitting $\mu_\Delta[\hat{W}_{i-1} F_l(P, K_{i-1})(j\omega)]$ by $|\hat{W}_i(j\omega)|$. The order of the H_2 controllers in the minimizing sequence can be chosen a priori according to the required closeness to the optimal μ solution. Once controller order is assigned, the order n_w of W_i for curve fitting can be determined accordingly.

The multiplication of $F_l(P, K_i)$ by a scalar function W_i can be considered as a shaping effect on the augmented plant P by noting that $W_i F_l(P, K) = F_l(P_i, K)$, where

$$P_i(s) = \begin{bmatrix} W_i(s) I_l & 0 \\ 0 & I_m \end{bmatrix} P$$

with l and m being the dimensions of the output vector z and the measurement vector y, respectively. In this way, a scalar weighting function, instead of a matrix-valued weighting function, can always be used effectively, regardless of the dimension of the augmented plant and the structure of the uncertainties.

In summary, the procedures of synthesizing the optimal μ controllers via H_2-based loop-shaping design involve only two steps: one step is the refinement of the frequency-dependent weight via $|\hat{W}_i(j\omega)| = \mu_\Delta[\hat{W}_{i-1} F_l(P, K_{i-1})(j\omega)]$; the other step is the refinement of the optimal H_2 controller via $K_i(s) = \arg\inf_K \|\hat{W}_i \times F_l(P, K)\|_2$.

Numerical Examples

Here we consider two examples. One is a single-input/single-output (SISO) disturbance rejection problem,[3] and the other is a multi-input/multi-output (MIMO) autopilot design problem.[6,11]

SISO Case: Disturbance Rejection

Referring to Fig. 2, the objective of this design problem is to synthesize controllers such that the disturbance output $y = [1 + (G + W_d\Delta_d)K]^{-1}d$ satisfies $\|W_p y\|_\infty < 1$ in the presence of the additive plant uncertainty $W_d\Delta_d$, where the magnitude W_d of the additive plant uncertainty, the disturbance output envelope W_p, and the nominal plant G are given,[3] respectively, as

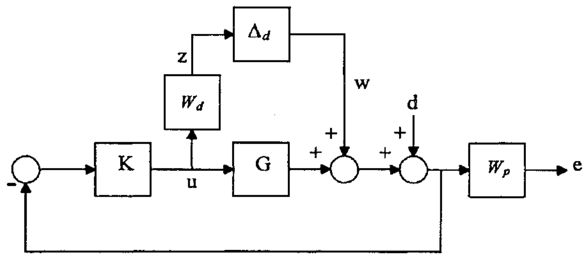

Fig. 2 Block diagram for the SISO design example.

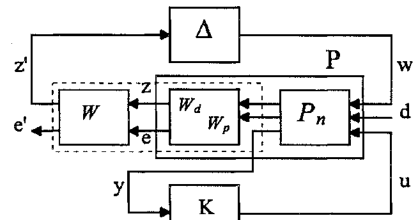

Fig. 3 Block diagram with auxiliary weighting function $W(s)$.

$$W_d(s) = \frac{0.2502s + 0.1256}{0.5s + 1}, \qquad W_p(s) = \frac{40.1606}{142.8571s + 1}$$
$$G(s) = \frac{0.5(1 - s)}{(s + 2)(s + 0.5)}$$

This robust performance problem can be recast to the μ-optimization problem:

$$\inf_{K \text{ stabilizing}} \sup_\omega \mu_\Delta[F_l(P, K)(j\omega)] \tag{32}$$

The optimal μ controller is to be solved by the H_2-based loop-shaping design introduced in the preceding sections. The interconnection of the frequency-dependent weight $W(s)$ with the original system is shown in Fig. 3. We will shape the weighting functions $W_d(s)$ and $W_p(s)$ such that the original μ-optimization problem can be converted to an equivalent weighted H_2-optimization problem with the reshaped weighting functions given by $W_i(s)W_d(s)$ and $W_i(s)W_p(s)$. Different orders of W_i and different number of iterations can be exploited to test the accuracy and the converging tendency of the proposed scheme. For instance, we use fourth-order $W_i(s)$ for curve fitting and three iterations have been performed to yield four optimal H_2 controllers: $K_i = \arg\inf_K \|W_i F_l(P, K)\|_2$, $i = 0, 1, 2, 3$ with $\deg[K_i(s)] = 11$. Numerical results for $\lambda_i = \|W_i F_l(P, K_i)\|_2$, $\sup_\omega \zeta_i(\omega) = \sup_\omega \mu_\Delta[F_l(P, K_i)(j\omega)]$, and $W_i(s)$, $i = 0, 1, 2, 3$ are shown in Table 1.

The plot of $\zeta_i = \mu_\Delta[F_l(P, K_i)(j\omega)]$ is shown in Fig. 4, where we can observe that $\mu_\Delta[F_l(P, K_3)(j\omega)]$ is almost all pass. Also shown in Fig. 4 is the frequency response $\mu_\Delta[F_l(P, K_{DK})(j\omega)]$ for which the controller $K_{DK}(s)$ is obtained from three D–K iterations with $\deg[K_{DK}(s)] = 11$. It is observed that $\sup_\omega \mu_\Delta[F_l(P, K_3)(j\omega)] = 1.0770$, whereas $\sup_\omega \mu_\Delta[F_l(P, K_{DK})(j\omega)] = 1.0689$. The effects of K_3 and K_{DK} are nearly indistinguishable. The resulting μ value is greater than 1, indicating that the robust performance requirement is not satisfied, and the magnitude W_d of the additive plant uncertainty needs to be shrunken, at least, by a ratio 1/1.0770.

During the iteration process, we use $\inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$ to approximate the structured singular value $\mu_\Delta(M)$; nevertheless, the convergence to the optimal μ controller is still guaranteed, since in this example $2s + f \leq 3$ is satisfied and the approximation is perfect. Therefore, if we fit W_i with higher degrees and increase the computational accuracy, K_i can be made arbitrarily close to the optimal μ solution. Indeed, for this simple SISO case, we have a closed-form expression for the structured singular value:

Table 1 H_2-based loop-shaping results

No.	λ_i	$\sup_\omega \zeta_i(\omega)$	$W_i(s)$
0	0.8414	1.5234	1
1	1.0376	1.1741	$\frac{0.00561s^4 + 12.78s^3 + 16{,}440s^2 + 26{,}010s + 16{,}510}{s^4 + 24{,}950s^3 + 44{,}910s^2 + 30{,}100s + 9109}$
2	1.0733	1.0927	$\frac{0.0095s^4 + 22.77s^3 + 32{,}200s^2 + 381{,}600s + 456{,}500}{s^4 + 69{,}700s^3 + 763{,}900s^2 + 592{,}300s + 254{,}900}$
3	1.0668	1.0770	$\frac{0.001328s^4 + 3.484s^3 + 5392s^2 + 19{,}130s + 16{,}000}{s^4 + 11{,}650s^3 + 34{,}000s^2 + 23{,}830s + 9135}$

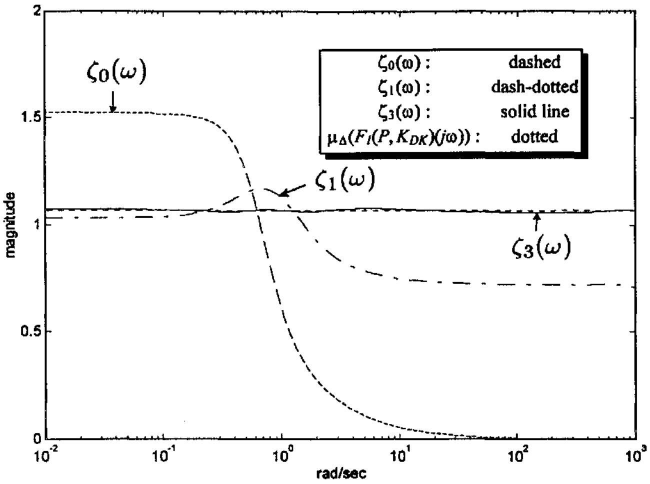

Fig. 4 Iteration results of the SISO design example.

$\mu_\Delta[M(j\omega)] = |M_{11}(j\omega)| + |M_{22}(j\omega)|$. Employing this formula in the iteration process could reduce the computational time greatly.

This example illustrates the fact, as proved earlier, that by recursive frequency shaping optimal H_2 controllers can converge graduately to the optimal μ controller.

MIMO Case: Autopilot Design

This section contains an example of μ synthesis as applied to the pitch-axis controller design of an experimental highly maneuverable airplane HIMAT.[6,11] The linearized model of HIMAT consists of four states: $x^T = (\delta v, \alpha, q, \theta)$, representing the forward velocity, angle of attack, pitch rate, and pitch angle, respectively. The control inputs are elevon deflection angle δ_e and the canard deflection angle δ_c. The variables to be measured are α and θ. The state–space nominal model for this two-input two-output plant is given by

$$G_0(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|cc} -0.023 & -37 & -19 & -32 & 0 & 0 \\ 0 & -1.9 & 0.98 & 0 & -0.41 & 0 \\ 0.012 & -12 & -2.6 & 0 & -78 & 22 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 57 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57 & 0 & 0 \end{array}\right]$$

A diagram for the closed-loop system, which includes the feedback structure of the plant and controller, and elements associated with the uncertainty models and performance objectives, is shown in Fig. 5. The control design objective is to determine a stabilizing K such that for all stable perturbation $\Delta_G(s)$, with $\|\Delta_G\|_\infty < 1$, the perturbed closed-loop system remains stable, and the perturbed weighted sensitivity transfer function $S(\Delta_G) = [I + G_0(I + \Delta_G W_d)K]^{-1}$ has $\|W_p S(\Delta_G)\|_\infty < 1$ for all such perturbations. The envelope W_p of the sensitivity function and the magnitude W_d of the multiplicative plant uncertainty, are given, respectively, as

$$W_p(s) = \frac{1.5}{s + 0.03}, \qquad W_d = \frac{100s + 10{,}000}{s + 10{,}000}$$

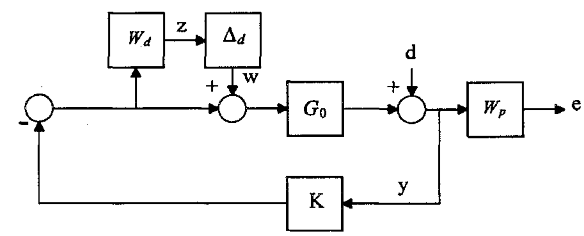

Fig. 5 Block diagram for the MIMO design example.

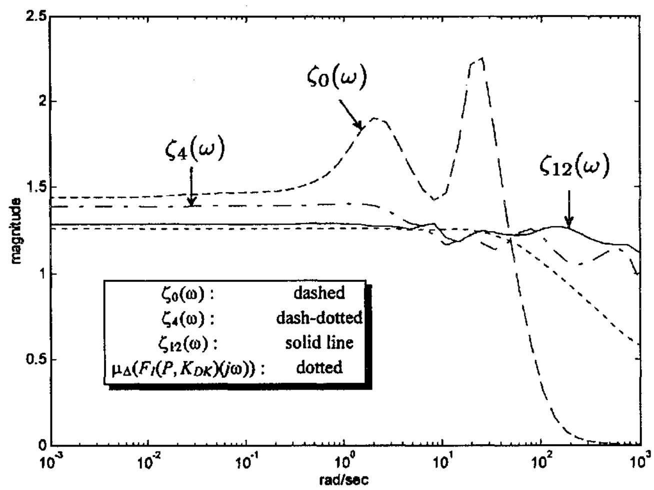

Fig. 6 Iteration results of the MIMO design example.

In terms of the structured singular value framework, these design objectives are satisfied if and only if $\inf_K \sup_\omega \mu_\Delta[F_l(P, K)(j\omega)] < 1$, where

$$F_l(P, K) = \begin{bmatrix} W_d T_0 & W_d K S_0 \\ W_p S_0 G_0 & W_p S_0 \end{bmatrix} \tag{33}$$

with S_0 and T_0 being the sensitivity and complementary sensitivity functions for the nominal plant G_0. The μ controller for this robust performance problem has been computed by D–K iteration.[6] Here, we will resolve the μ-optimization problem via a sequence of optimal H_2 controllers $K_i(s)$. The frequency response of $\mu_\Delta[F_l(P, K_i)(j\omega)]$ is depicted in Fig. 6, where 12 iterations have been performed. It can be seen that $\mu_\Delta[F_l(P, K_{12})(j\omega)]$ is nearly uniform over the low-frequency range, being close to the μ solution obtained by D–K iteration. The rolloff tendency at high frequency is primarily caused by the roundoff errors and the truncation errors in calculating the structured singular value. If we increase the computational accuracy, the uniform portion of $\mu_\Delta[F_l(P, K_i)(j\omega)]$ can be extended to cover a wider frequency range. For comparison purposes, we have chosen the orders of W_i such that $\deg(K_i) = \deg(K_{DK})$, and the number of iterations involved in obtaining K_i and K_{DK} is kept the same. The weighting functions $W_d(s)$ and $W_p(s)$ before and after shaping by the frequency-dependent weights $W_i(s)$ are shown in Figs. 7 and 8. Since weighted H_2-optimization problems can be solved by the LQG/LTR design procedure,[11] we

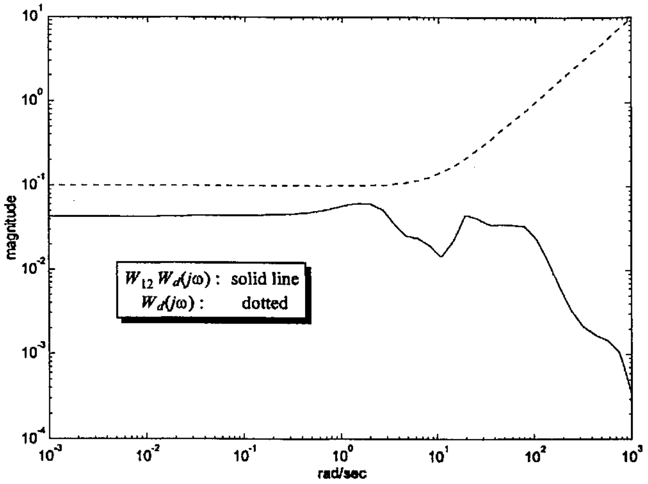

Fig. 7 Reshaping of $W_d(s)$ by $W_{12}(s)$.

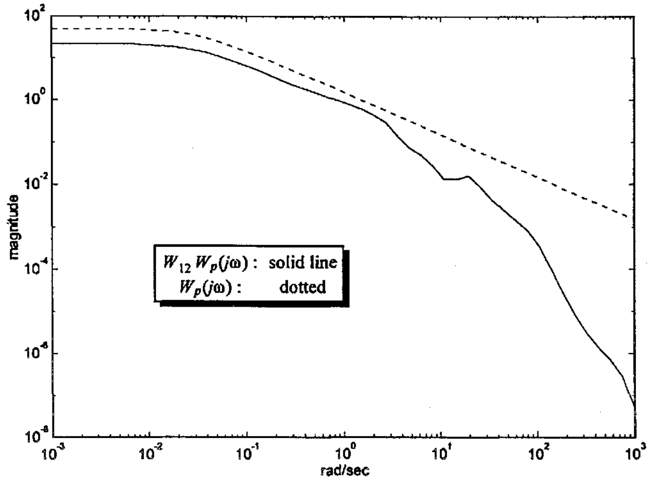

Fig. 8 Reshaping of $W_p(s)$ by $W_{12}(s)$.

thus can implement the H_2-based loop-shaping algorithm in terms of LQG/LTR to obtain the optimal μ controllers.

There are two main differences between the H_2-based μ-synthesis technique and the H_∞-based μ-synthesis technique (D–K iteration).

1) The convergence of the H_2-based μ-synthesis procedures to the optimal μ controller is theoretically guaranteed, provided that the structured singular value of a constant matrix with any type of uncertainty structures can be obtained perfectly. The D–K iteration is not guaranteed to converge to the minimum of μ, even if $2s + f \leq 3$.

2) The optimal H_2 controller can be obtained in closed form,[12] whereas the optimal H_∞ controller can only be obtained iteratively by using the γ-iteration technique.[12] For example, if the optimal H_∞ controller is computed to an accuracy 10^{-4}, the typical number of γ iterations required is about an order of 10. Consequently, at every iteration, the H_2-based μ-synthesis technique is required to solve the two algebraic Riccati equations[12] (ARE) only once, but the H_∞-based μ-synthesis technique needs to solve the two ARE 10 times. In the two examples, the number of the iterations of the two approaches are kept the same, and comparable accuracy results; however, the number of times ARE are solved for the proposed algorithm is far below that for D–K iteration.

Conclusions

We have derived a sufficient condition for the existence of the optimal μ controllers by showing that the μ-optimization problem is equivalent to the weighted H_2 optimization problem. With this approach, we have verified the possibility that conventional LQG controllers, with iterative tuning of frequency-dependent weights, can become optimal μ controllers. By providing a systematic methodology of determining the frequency-dependent weights via the H_2-based loop-shaping procedures, we have derived, both theoretically and numerically, LQG controllers converging to the optimal μ controllers. The proposed algorithm yields a comparable accuracy with D–K iteration, but with reduced computational efforts.

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